

Reconstructing a general inflationary action

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If inflation is to be considered in an unbiased way, as possibly originating from one of a wide range of underlying theories, then observations need not be simply applied to reconstructing the inflaton potential, $V(\phi)$, or a specific kinetic term, as in DBI inflation, but rather to reconstruct the inflationary action in its entirety. We discuss the constraints that can be placed on a general single field action from measurements of the primordial scalar and tensor fluctuation power spectra and non-Gaussianities. We also present the flow equation formalism for reconstructing a general inflationary Lagrangian, $\mathcal{L}(X, \phi)$, with $X = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$, in a general gauge, that reduces to canonical and DBI inflation in the specific gauge $\mathcal{L}_X = c_s^{-1}$.

I. INTRODUCTION

The Cosmic Microwave Background (CMB) [1, 2, 3, 4] is now measured with exquisite precision from horizon scales down to a few arc minutes angular resolution. In combination with large scale structure surveys [5, 6, 7], this allows the primordial spectrum of fluctuations to be characterized in fine detail [8, 9, 10, 11].

There has been significant effort to relate the observed primordial spectrum of fluctuations to the underlying theory that seeded them. In the context of slow roll inflation [12, 13, 14], taking a specific potential and comparing it to data is a sensible approach to assess if the theory is consistent (see for example [8, 15]). An alternative application of the data, however, is to invert this process, and reconstruct what we can know about the underlying theory [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34].

With the introduction of a broader array of inflationary theories, for example arising out of the Dirac-Born-Infeld action [35] or from k-inflation [36, 37], the interpretation of observations has necessarily extended beyond focusing only on the inflationary potential to including information about the form of the Lagrangian kinetic term. Recently, cosmological constraints on brane inflation models have been studied both in the context of specific models [38, 39, 40] and in more model-independent studies [41]. If inflation is to be considered without theoretical bias, then the objective must not be to simply reconstruct the inflaton potential, or a specific kinetic term, but rather to reconstruct what observations tell us quantitatively about the effective inflaton action in its entirety. In this paper, we develop a formalism for such a general inflationary reconstruction in the context of single field models, and present explicit analytic techniques for action reconstruction.

In the usual potential reconstruction formalism, the inflationary observables as a function of scale can be mapped to the behavior of the inflationary potential

$V(\phi)$ as a function of the inflaton field ϕ . If for example the scalar spectral index $n_s(k)$ can be extracted exactly from data, the shape of the potential $V(\phi)$ can be deduced in the usual formalism if a reheating scenario is fixed. However, the reconstruction of the entire action, including the possibility of non-minimal kinetic terms, is harder as the action is now a functional of two independent functions, $X \equiv (\partial\phi)^2/2$ and ϕ . To fix this new functional degree of freedom, in principle a continuous set of independent data analogous to $n_s(k)$ is needed (i.e. an infinite number of observables). Although daunting, the formalism that we present may be used as a starting point to connect cosmological data to high energy theories which may have other phenomenological, theoretical, and aesthetic constraints.

The analytic form of the non-minimal kinetic actions consistent with data can be written in a surprisingly simple form given in section VI by Eq. (85). Each consistent action is simply a manifold parameterized by X and ϕ satisfying certain derivative conditions on a one dimensional submanifold which represents the data. Furthermore, using the Hamilton-Jacobi formalism, we extend the inflationary flow parameter approach to describe the evolutionary trajectories of general actions. This involves introducing three hierarchies of flow parameters to describe the evolution of a general action without using the specific restriction of field redefinition used in canonical and DBI inflation. These equations hold for all single field inflationary scenarios, whether or not slow roll conditions are met.

The importance of including kinetic terms in the inflationary reconstruction program cannot be overemphasized in light of recent theoretical and expected experimental advances. Inflationary models with non-minimal kinetic terms are able to produce large non-Gaussian behavior for the curvature perturbations without ruining other inflationary observables [42] and predictions of non-Gaussian signatures for specific models have been established, for example in DBI inflation [40, 43, 44, 45, 46]. Indeed, the search for

such non-Gaussian effects is one of the primary current activities in observational cosmology, for example [11, 47, 48]. Non-Gaussianity detections open up the possibility of establishing which non-minimal kinetic terms may exist for inflationary models. The formalism that we present here will be useful for this purpose.

The explanation of any future (or current) observations of non-Gaussianities can also be checked in the context of single field inflation through the attendant modification of the tensor spectral index consistency relationship [37]. The latter can be deduced experimentally from observations of tensor perturbations implied by CMB B-mode polarization measurements. However, one advantage of emphasizing the non-Gaussianity connection with non-minimal kinetic terms is that the possibility of a large non-Gaussian contribution is generically independent of the single field paradigm. On the other hand, the tensor spectral index consistency relationship changes for multi-field inflationary models.

The order of our presentation will be as follows. In section II, we review and clarify the physics of how non-minimal kinetic terms contribute to non-Gaussianities, whose possible future observation is one of the strongest motivations for developing the action reconstruction formalism. In III we outline the general equations for the background evolution. We discuss the conditions for slow roll inflation in a general action in section IV. In V we summarize how the generalized flow parameters relate to the properties of the primordial power spectrum and discuss how properties of a general action can be distinguished from canonical inflation using cosmological observations. In section VI we establish how the general action can be reconstructed from measurements of the lowest order flow parameters in the slow roll regime, and in VII we extend the inflationary flow parameters [49] to describe a general inflationary action. In VIII we summarize our findings and discuss their implications.

Throughout this paper with the exception of section VI, we use the usual reduced Planck scale conventions of $M_{pl}^2 = (8\pi G)^{-1} \approx (2.4 \times 10^{18} \text{GeV})^2$. In section VI, we will use geometricized units and set $M_{pl} = 1$ for simplicity in notation.

II. NON-GAUSSIANITY AND NON-MINIMAL KINETIC TERMS

Although there have been many previous works [16, 17, 18, 19, 20, 21, 22, 23, 24, 25] on inflationary potential reconstruction, there are relatively fewer works on trying to reconstruct kinetic terms [41]. As explained in the introduction, one of the main motivations for focusing on non-minimal kinetic terms is its importance to non-Gaussian observables, whose search is an active area of research in observational cosmology. In this section, we explain how non-

minimal kinetic terms can generate observable non-Gaussian statistics. Most of this section is devoted to summarizing and clarifying the literature which is particularly relevant for this paper.

All field correlation functions of a *non-interacting* field theory can be reduced to the information in the two-point correlation function similar to the statistics of a classical Gaussian random variable. During slow roll inflation, the energy density fluctuations of the inflaton ϕ (the dominant energy component) are approximately

$$\delta\rho_\phi(x) \sim V'(\phi_0)\delta\phi(x) \quad (1)$$

where $V(\phi)$ is the inflaton potential, ϕ_0 is the classical time dependent background homogeneous inflaton field, and $\delta\phi(x)$ is the quantum fluctuating inflaton field degree of freedom. Hence, if $\delta\phi$ fluctuations (which eventually decohere to become classical) can be described by a non-interacting field theory, then the statistics of $\delta\rho_\phi$ will also be Gaussian since by the linear relationship given in Eq. (1), it inherits the statistics of $\delta\phi$.

In slow roll inflationary theories with minimal canonical kinetic terms, the inflaton field still interacts non-trivially with gravity, leading to non-Gaussian statistics of $\delta\rho_\phi$. However, because the energy density fluctuations are small, the gravity-mediated self-interactions are typically small. Furthermore, the slow roll constraints also phenomenologically forces the coupling constants in the self-interaction terms of the potential to be small, suppressing non-gravity-mediated self-interactions. One typical characterization of the non-Gaussian statistics is the 3-point function

$$\langle \zeta(\tau, \vec{k}_1) \zeta(\tau, \vec{k}_2) \zeta(\tau, \vec{k}_3) \rangle = (2\pi)^7 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \times [P^\zeta(k_1 + k_2 + k_3)]^2 \frac{\mathcal{A}(\vec{k}_1, \vec{k}_2, \vec{k}_3)}{\prod_i k_i^3} \quad (2)$$

where \mathcal{A} is a smooth function with dimension $[k]^3$, ζ is the scalar perturbation which in the $\delta\phi = 0$ gauge parameterizes the spatial metric as $\exp(2\zeta)|d\vec{x}|^2$, τ is conformal time when all the scales are far outside of the horizon during inflation, and P_k^ζ is the two-point function power spectrum.[65] To linear order, the scalar perturbation reduces to the linearly gauge invariant function $\zeta = -\Psi - \frac{H}{\dot{\phi}_0}\delta\phi$, where Ψ is scalar perturbation appearing in the line element $(dt^2(1+2\Psi))$ and H is the expansion rate. The literature often characterizes the amplitude \mathcal{A} at either the $\vec{k}_1 = \vec{k}_2 = \vec{k}_3$ limit (equilateral triangle), or $|\vec{k}_1| \ll |\vec{k}_2|, |\vec{k}_3|$ limit (squeezed or local limit). In each of these cases, a dimensionless quantity f_{NL} can be defined by the relation [45]

$$\mathcal{A}(\vec{k}_1, \vec{k}_2, \vec{k}_3) \equiv -\frac{3}{10} f_{NL}^{\text{equal or local}} \sum_i k_i^3 \quad (3)$$

where the definition is motivated by the characterization of non-Gaussianities by a non-general ansatz $\zeta = \zeta_G - \frac{3}{5}f_{NL}^{\text{equil}}$ or local $(\zeta_G^2 - \langle \zeta_G^2 \rangle)$ which is valid only when the non-Gaussian variable ζ is related to the Gaussian variable ζ_G by a local field redefinition.

Having a f_{NL}^{equil} or local > 0 in the sign convention of Eq. (3) corresponds to having more hot spots in the CMB anisotropies compared to the case with $f_{NL} = 0$. To see this, note that the observed anisotropies on large scales is approximately $\frac{\Delta T}{T} \approx -\frac{1}{5}\zeta$ due to Sachs-Wolfe effect. Hence, the temperature one point function $P(\Delta T/T)$ should behave approximately as

$$\ln P \propto \left[\frac{\Delta T}{T} - 3f_{NL} \left(\left(\frac{\Delta T}{T} \right)^2 - \left\langle \left(\frac{\Delta T}{T} \right)^2 \right\rangle \right) \right]^2, \quad (4)$$

which makes the probability of having $\Delta T/T$ larger than the standard deviation a bit higher. Note that the sign convention of [45, 46] is opposite to the sign convention of [48]. Furthermore, the non-zero value of f_{NL} measured by [48] is in the squeezed limit of $k_1 \ll k_2, k_3$ which is less sensitive to the non-minimal kinetic term as pointed out by several papers (e.g. see [43, 44, 45, 46]). To leading order, f_{NL} can be related to the scalar perturbation spectral index as

$$f_{NL}^{\text{local}} \sim (n_s - 1) \quad (5)$$

which is suppressed (as these are proportional to the slow roll parameters) with a negative sign in the current sign convention if $n_s < 1$. Hence, it is interesting that the result of [48] is *not* likely to be explained by something like DBI inflation or more generally, by non-minimal kinetic term effect only. For slow roll inflationary models, this small f_{NL} proportional to the slow roll parameter is generic [46].

One idea to make f_{NL} large from non-gravitational self-interactions that people did not pay much attention to before [42] was that generically self-interactions can be made large without preventing inflation if the self-interactions come from non-minimal kinetic terms. From an intuitive point of view, one sees that if the inflaton Lagrangian has the form

$$\mathcal{L}_{\text{intuition}} = f((\partial\phi)^2, \phi)(\partial\phi)^2 - m^2\phi^2 \quad (6)$$

where m is a mass parameter and $f(a, b)$ is a function which has a large numerical value, say $Z \gg 1$, along a particular classical solution, then by redefining the field to be $\tilde{\phi} \equiv \sqrt{Z}\phi$, we have numerically

$$\mathcal{L}_{\text{intuition}} \sim (\partial\tilde{\phi})^2 - \frac{m^2}{Z}\tilde{\phi}^2. \quad (7)$$

This makes the effective potential even flatter than the situation in which Z was of order 1, which in turn helps in meeting the phenomenological inflationary conditions. At the same time, if $Z = f((\partial\phi)^2, \phi)$ is large, then there are non-renormalizable

self-interactions of ϕ in Eq. (6) that are large, and hence the expected non-Gaussianities can be large without spoiling inflation.

Therefore, one key to obtaining large non-Gaussianities in inflationary models with a single scalar field is to consider modifications of the kinetic term. To see how f_{NL} can be related to the nonrenormalizable interactions appearing in the kinetic sector, consider a dimension 8 non-renormalizable interaction of the form

$$S_{\text{int}} = \int d^4x \sqrt{g} \frac{c}{\Lambda^4} (\partial_\mu \phi \partial^\mu \phi)^2. \quad (8)$$

After expanding ϕ as $\phi_0(t) + \delta\phi(x)$, we see that Eq. (8) contains a cubic self-interaction

$$S_{\text{int}} \ni \int d^4x a^3 \frac{4c}{\Lambda^4} \dot{\phi}_0(t) \delta\dot{\phi}^3(x) \quad (9)$$

where the dot denotes the partial derivative with respect to a comoving observer's proper time and $\phi_0(t)$ is governed by the quadratic Lagrangian

$$S_2 = \int d^4x \sqrt{g} \left[\frac{1}{2} (\partial_\mu \phi \partial^\mu \phi) - V(\phi) \right]. \quad (10)$$

For the present discussion, we will assume that $V(\phi)$ energy density and pressure are dominant during inflation. It is interesting to note that the cubic derivative interaction of Eq. (9) is induced by having *both* a dimension 8 short distance operator (Eq. (8)) and a nonvanishing time dependent background field $\phi_0(t)$. The second condition is required because local Lorentz invariance forces Eq. (8) to have a Z_2 symmetry which needs to be spontaneously broken by $\phi_0(t)$ to obtain a cubic interaction.

If the scalar metric fluctuation can be neglected, then the dominant contribution to the 3-point function would simply come from Eq. (9). However, as is well known by now ([44, 45]), the scalar metric perturbations induce significant contributions to $\langle \zeta \zeta \zeta \rangle$ proportional to c in Eq. (8). To account for the scalar metric perturbations, it is often more convenient to choose the foliation of spacetime with spacelike 3-surfaces in which $\delta\phi = 0$ and use the ADM formalism to construct the interacting Lagrangian and Hamiltonian [46] in terms of the metric component $\exp(2\zeta)$ characterizing 3-metric as $\exp(2\zeta)d|\vec{x}|^2$. Explicitly, the leading interaction Lagrangian can be written as [44]

$$\mathcal{L}_I \sim a^3 \epsilon u \left[\frac{-2}{3} \frac{\dot{\zeta}^3}{H} + 8a^2 \dot{\zeta}^2 \partial^{-2} \dot{\zeta} \right] \quad (11)$$

where $u = \frac{-8c\dot{\phi}_0^2}{\Lambda^4}$ for the case of Eq. (8). The second term (whose peculiar non-local form comes from solving the non-local constraint equations of gravity) turns out to dominate in contribution to $\langle \zeta \zeta \zeta \rangle$ over the local interactions represented in the first term in

the limit that the slow roll parameters vanish. This indicates that the metric perturbations cannot be neglected in computing the 3-point functions for non-minimal kinetic terms. What is intriguing about this is that non-Gaussianities may in fact be a sensitive probe of gravity. Although we will leave investigations of this issue to a future work, it is interesting to note that the modifications of gravity proposed by [50, 51] directly changes the gravitational constraint equations which the scalar metric perturbations are sensitive to.

The 3-point function is computed perturbatively as

$$\langle \mathcal{O}(t) \rangle = \langle e^{i \int_{t_0}^t dt' H_I(t')} \mathcal{O}_{int}(t) e^{-i \int_{t_0}^t dt' H_I(t')} \rangle \quad (12)$$

$$\approx \langle \mathcal{O}_{int} \rangle + i \int_{t_0}^t dt' \langle [H_I(t'), \mathcal{O}_{int}(t)] \rangle \quad (13)$$

where $\mathcal{O} = \zeta(t, \vec{k}_1) \zeta(t, \vec{k}_2) \zeta(t, \vec{k}_3)$, H_I is the interacting Hamiltonian ($H_I = - \int d^3x \mathcal{L}_I$), and the first term is vanishing for our observable. Hence, noting that in the spatially flat gauge, $\zeta \sim \frac{-H}{\dot{\phi}_0} \delta\phi$, and near the slow roll limit $\dot{\phi}_0 \sim \sqrt{2\epsilon_V} H M_{pl} \text{sgn}(\dot{\phi}_0)$ (where $\epsilon_V \equiv M_{pl}^2 (V'(\phi)/V(\phi))^2/2$), the leading effect of the non-minimal kinetic term on the 3-point function can be represented by the diagram in Fig. 1. Because the background spacetime has spatial translational invariance, there will still be an overall 3-momentum conservation $(2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)$ in the computation. However, because of the propagator being in dS spacetime patch which is not time translationally invariant, the time integral will not conserve $\sum_i |\vec{k}_i|$ but will instead lead to a factor of $1/H$. Since the vertex of the first diagram in Fig. 1 can be read off from Eq. (9) as $\frac{c\dot{\phi}_0}{\Lambda^4}$, we can estimate

$$\begin{aligned} \langle \zeta(t, \vec{k}_1) \zeta(t, \vec{k}_2) \zeta(t, \vec{k}_3) \rangle &\sim (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \times \\ &\frac{1}{H} \times \frac{c\dot{\phi}_0}{\Lambda^4} \times \left(\frac{H}{\dot{\phi}_0}\right)^3 \times (H^2)^3 h(\vec{k}_1, \vec{k}_2, \vec{k}_3) \quad (14) \\ &= (2\pi)^7 \delta^{(3)} \left(\sum_i \vec{k}_i \right) \frac{c\dot{\phi}_0^2}{\Lambda^4} \left[\left(\frac{H}{\dot{\phi}_0} \frac{H}{2\pi} \right)^2 \right]^2 h(\vec{k}_1, \vec{k}_2, \vec{k}_3) \quad (15) \end{aligned}$$

where $h(\vec{k}_1, \vec{k}_2, \vec{k}_3)$ is a Fourier transform related kinematic function which scales as $1/k^6$ in the equilateral triangle limit of $|\vec{k}_1| = |\vec{k}_2| = |\vec{k}_3| = k$, and the other factors in the first equation can be explained as follows. The factor of $1/H$ comes from the integral $\int dt$ (the effective interaction having a time scale of order $1/H$, and this is what results from the nonconservation of $\sum_i |\vec{k}_i|$), the factor of $c\dot{\phi}_0/\Lambda^4$ comes from the interaction vertex, $(H/\dot{\phi}_0)^3$ comes from the relationship $\zeta \sim \frac{-H}{\dot{\phi}_0} \delta\phi$, and $(H^2)^3$ comes from each of the external $\langle \delta\phi \delta\phi \rangle$ propagators in Fig. 1 being proportional to H^2 (the well known massless dS propagator scaling). The second line follows from trying to ex-

press the result as Eq. (2), where

$$P_k^\zeta \approx \left(\frac{H}{\dot{\phi}_0} \frac{H}{2\pi} \right)^2 \quad (16)$$

to leading slow roll and non-minimal kinetic term vertex order.

Comparing Eq. (15) with Eq. (3) and Eq. (2), we find for equilateral momentum triangle configuration

$$f_{NL}^{\text{equil}} \sim c \frac{\dot{\phi}_0^2}{\Lambda^4}, \quad (17)$$

up to a minus sign which we cannot predict with the current detail of estimation, which leaves out the gravitational effects (e.g. the second diagram of Fig. 1). [66] The inclusion of the gravitational effects [44, 45] yields

$$f_{NL}^{\text{equil}} \approx 0.28 \frac{2X \mathcal{L}_{XX}}{\mathcal{L}_X} \quad (18)$$

which for Eq. (8) yields

$$f_{NL}^{\text{equil}} \approx 0.28 \frac{8c\dot{\phi}_0^2}{\Lambda^4} \quad (19)$$

in agreement with Eq. (17).

Before closing this section, we would like to also comment that Eq. (17) can be rewritten in terms of the potential slow roll parameters as

$$f_{NL}^{\text{equil}} \sim c\epsilon_V \frac{H^2}{\Lambda^2} \frac{M_{pl}^2}{\Lambda^2}. \quad (20)$$

This expression is interesting because although the dimension 8 operators of the form of Eq. (8) are generically expected to exist in conventional effective field theories with $c \sim \mathcal{O}(1)$ because Λ then is the cutoff scale, the validity of the effective field theory description requires

$$H^2 M_{pl}^2 < \Lambda^4. \quad (21)$$

Eq. (20) would then imply that the f_{NL} contribution from perturbatively treated non-minimal kinetic operators would be suppressed by ϵ_V in a typical effective field theory. However, there are apparently situations such as in DBI inflationary models in which $\frac{2X \mathcal{L}_{XX}}{\mathcal{L}_X}$ can be large yet a sensible effective field theory description exists [52, 53]. Such scenarios would still give a large value for f_{NL}^{equil} due to non-minimal kinetic term interactions.

III. BACKGROUND EVOLUTION

Consider a general action of a single scalar field with a Lagrangian of the form $\mathcal{L}(X, \phi)$ where $X =$

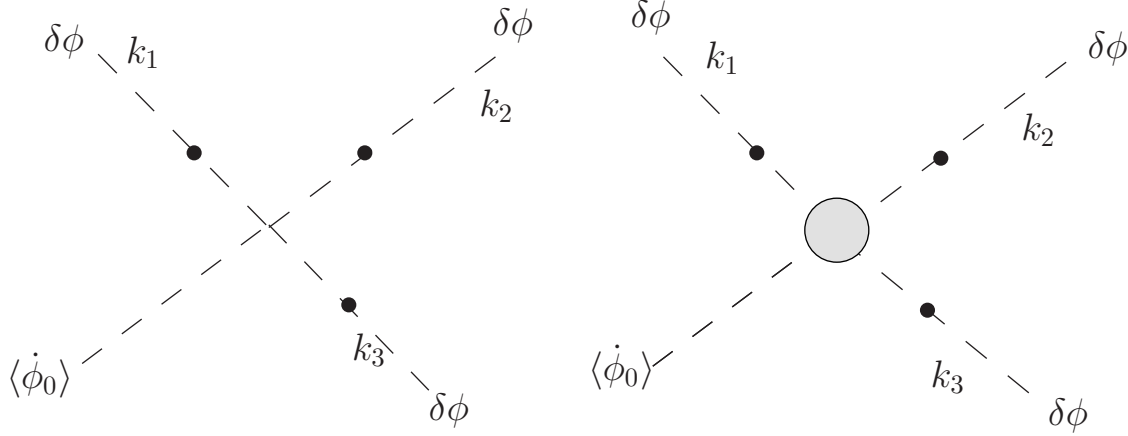


FIG. 1: Dimension 8 kinetic operator interaction of $\delta\phi$ contribution to the 3-point function of $\langle \zeta(t, \vec{k}_1) \zeta(t, \vec{k}_2) \zeta(t, \vec{k}_3) \rangle$. The small dots indicates the fact that the $\delta\phi$ propagator is a dS propagator (i.e. there is an interaction with the background classical homogeneous gravitational field leading to a time dependent mass). The blob on the right indicates that it is an interaction term (partly non-local) arising from the presence scalar metric fluctuations.

$\frac{1}{2} \partial_\mu \phi \partial^\mu \phi$ is the canonical kinetic term. One can describe the ϕ field by a hydrodynamical fluid in the following way:

$$T_{\mu\nu} = (p + \rho) u_\mu u_\nu - p g_{\mu\nu}, \quad (22)$$

where

$$p(X, \phi) \equiv \mathcal{L}(X, \phi), \quad (23)$$

$$\rho(X, \phi) \equiv 2X \mathcal{L}_X - \mathcal{L}(X, \phi), \quad (24)$$

$$u_\mu \equiv \frac{\partial_\mu \phi}{\sqrt{2X}}, \quad (25)$$

where $\mathcal{L}_X \equiv \partial \mathcal{L} / \partial X$. In the homogenous limit that $X = \frac{1}{2} \dot{\phi}^2$ Eq. (24) simplifies to

$$\rho(X, \phi) = \dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L}. \quad (26)$$

In this paper we assume that null energy condition $\rho + p > 0$, is satisfied, such that

$$2X \frac{\partial \mathcal{L}}{\partial X} > 0. \quad (27)$$

The Friedmann, acceleration and continuity equations for the background are

$$H^2 = \frac{1}{3M_{pl}^2} (2X \mathcal{L}_X - \mathcal{L}), \quad (28)$$

$$\frac{\ddot{a}}{a} = -\frac{1}{3M_{pl}^2} (X \mathcal{L}_X + \mathcal{L}), \quad (29)$$

$$\dot{\rho} = -3H(\rho + p), \quad (30)$$

where $a(t)$ is the scale factor, H is Hubble's constant $\equiv \dot{a}/a$. Accelerative expansion requires

$$0 < \frac{X \mathcal{L}_X}{-\mathcal{L}} < 1, \quad (31)$$

with $\mathcal{L} < 0$. The resulting equation of motion for the scalar field is

$$\dot{X} = \frac{\sqrt{2X} c_s^2}{\mathcal{L}_X} \left(\mathcal{L}_\phi - 2X \mathcal{L}_{X\phi} - 3H \sqrt{2X} \mathcal{L}_X \right), \quad (32)$$

where throughout we choose the sign of $\sqrt{2X}$ to be same as $\dot{\phi}$. c_s^2 is defined as

$$c_s^2 \equiv \frac{p_X}{\rho_X} = \left(1 + 2 \frac{X \mathcal{L}_{XX}}{\mathcal{L}_X} \right)^{-1}. \quad (33)$$

As we will see in the next section, it turns out to be the adiabatic sound speed for inhomogeneities. Requiring $c_s^2 \leq 1$ and the positivity of \mathcal{L}_X , giving $c_s^2 > 0$ from Eq. (27) yields,

$$\mathcal{L}_{XX} > 0. \quad (34)$$

Combining Eq. (28) and Eq. (30) we can write the kinetic variable as a function H and \mathcal{L}_X ,

$$\sqrt{2X} = -\frac{2M_{pl}^2}{\mathcal{L}_X} H', \quad (35)$$

where a prime denotes a total derivative with respect to ϕ ,

$$H' \equiv \frac{dH}{d\phi} = \frac{1}{\sqrt{2X}} \frac{dH}{dt}, \quad (36)$$

where

$$\frac{d}{dt} = \dot{X} \frac{\partial}{\partial X} + \sqrt{2X} \frac{\partial}{\partial \phi}. \quad (37)$$

Using Eq. (32), we can therefore describe all time derivatives in terms of partial derivatives with respect to ϕ and X .

Notice that if $\mathcal{L}(X, \phi)$ is known, $\mathcal{L}_X(X, \phi)$ can be derived and inserted in Eq. (35) to solve for $X(H', \phi)$, which then can be substituted back in Eq. (28) to obtain a nonlinear first order differential equation for $H(\phi)$ which similar to canonical actions corresponds to the Hamilton-Jacobi (HJ) equation for the general action [54]

$$3M_{pl}^2 H^2(\phi) = \frac{4M_{pl}^4 H'^2}{\mathcal{L}_X(X(H', \phi), \phi)} - \mathcal{L}(X(H', \phi), \phi). \quad (38)$$

We can rewrite the HJ equation in terms of a new parameter ϵ ,

$$3M_{pl}^2 H^2 \left(1 - \frac{2\epsilon}{3}\right) = -\mathcal{L}, \quad (39)$$

where

$$\epsilon \equiv \frac{3(\rho + p)}{2\rho} = -\frac{\dot{H}}{H^2} \quad (40)$$

and it can also be written in following formats in terms of parameters in the action,

$$\epsilon = \frac{3}{2 - \frac{\mathcal{L}}{X\mathcal{L}_X}} = \frac{2M_{pl}^2}{\mathcal{L}_X} \left(\frac{H'}{H}\right)^2. \quad (41)$$

The physical relevance of ϵ is more clearly seen given

$$\frac{\ddot{a}}{a} = (1 - \epsilon)H^2, \quad (42)$$

which implies that the accelerative expansion condition Eq. (31) can also be written for H as $0 < \epsilon < 1$.

To design a successful inflationary scenario, it is necessary to first address the flatness and horizon problem, for which it suffices to have [55]

$$\tilde{N} \equiv \ln \left| \frac{a(t_{en})H(t_{en})}{a(t_{in})H(t_{in})} \right| > \ln \left((1 + z_{eq})^{-1/2} \frac{T_{re}}{T_0} \right), \quad (43)$$

where T_0 , T_{re} and z_{eq} are the CMB temperature today, the reheating temperature, and the redshift at time of matter-radiation equality, respectively. The left hand side describes the logarithmic shrinkage of Hubble radius in the comoving frame. To be consistent with observations, the reheating temperature has to be above nucleosynthesis scales which yields $\tilde{N} \geq 24$, but if one assumes that a reheating temperature is as high as the GUT scale then a larger lower limit, $\tilde{N} \geq 80$, is obtained.

A commonly used measure of inflation is the number of e-folds of inflation, N_e , defined as

$$\begin{aligned} N_e &\equiv \ln \frac{a(t_{en})}{a(t_{in})} = - \int_{t_{en}}^{t_{in}} H dt \\ &= \frac{1}{M_{pl}} \int_{\phi_{in}}^{\phi_{en}} \sqrt{\frac{\mathcal{L}_X}{2\epsilon}} d\phi, \end{aligned} \quad (44)$$

in which t_{in} and t_{en} are the start and end of inflation, and we choose N_e to increase as one goes backwards in time from the end of inflation i.e $dN_e = -H dt$.

If H is changing slowly during inflation, then $N_e \sim \tilde{N}$ and the constraint on the Hubble radius shrinkage can be satisfied simply by requiring $N_e > \ln \left((1 + z_{eq})^{-1/2} \frac{T_{re}}{T_0} \right)$. In general, however, once one enforces the null energy condition $\rho + p > 0$, since \dot{H} is negative, inevitably N_e is $\tilde{N} + \ln \frac{H_{in}}{H_{en}}$, and hence is larger than \tilde{N} :

$$N_e > \frac{\tilde{N}}{1 - \epsilon_{min}}, \quad (45)$$

which implies that for scenarios in which ϵ is not close to zero one must obtain significantly larger number for e-folding to solve the horizon problem. Keep in mind that imposing a higher reheating temperature, and requiring the initial condition $\rho_{in} < M_{pl}^4$ at the same time, will only restrict $\ln[H(t_{in})/H(t_{en})] < 38$, and only marginally constrain ϵ to be about 10% less than one. Regardless of the null energy condition, meeting the observable constraint Eq. (43) guarantees that $\ddot{a}(t) > 0$ at least for some time even if a scenario is designed to avoid a large number of e-foldings.

For a general action, one can define two further dynamical parameters in addition to ϵ that (as will be shown in the next section) control the slow roll regime and are directly measurable by observations:

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} \quad (46)$$

$$\kappa \equiv \frac{\dot{c}_s}{Hc_s}. \quad (47)$$

These parameters are independent of the choice of scalar field definition (the field gauge choice) in the action.

IV. SLOW ROLL CONDITIONS FOR A GENERAL ACTION

In the case of a general inflation model the term “slow roll” can be ambiguous. Here we ensure that slow roll is independent of a scalar field definition, and purely relates to the gauge invariant flow parameters:

$$\epsilon, \eta, \kappa, \eta_N, \kappa_N \dots \ll 1, \quad (48)$$

where $\eta_N \equiv d\eta/dN_e$ etc.. These parameters are dependent upon \mathcal{L} and gauge invariant combinations of X and derivatives of \mathcal{L} with respect to X and ϕ .

In this paper we do not establish whether particular actions are able to realize slow roll inflation. However, in the following sections, we do consider the implications for evolution if slow roll behavior is satisfied. From Eq. (41) we can see that $\epsilon \ll 1$ implies,

$$\frac{X\mathcal{L}_X}{-\mathcal{L}} \ll 1. \quad (49)$$

Combined $\epsilon \ll 1$ and $\eta \ll 1$ suggest that

$$\eta - 2\epsilon = \frac{X\dot{\mathcal{L}}_X}{HX\mathcal{L}_X} \ll 1, \quad (50)$$

while $\kappa \ll 1$ implies,

$$\frac{1 - c_s^2}{2} \left| \frac{(X^2\dot{\mathcal{L}}_{XX})}{HX^2\mathcal{L}_{XX}} - \frac{(X\dot{\mathcal{L}}_X)}{HX\mathcal{L}_X} \right| \ll 1. \quad (51)$$

Note that, as such, this ‘slow roll’ condition allows for scenarios such as ultra-relativistic DBI inflation in which $c_s^2 \ll 1$ if

$$\left| \frac{(X^2\dot{\mathcal{L}}_{XX})}{HX^2\mathcal{L}_{XX}} \right| \ll 1. \quad (52)$$

V. OBSERVATIONAL CONSTRAINTS ON SLOW ROLL PARAMETERS FROM THE PRIMORDIAL SPECTRUM

In the absence of anisotropic stress in energy momentum tensor at linear order, we can write the metric in the longitudinal gauge as [56]:

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)a^2(t)\gamma_{ij}dx^i dx^j. \quad (53)$$

Just as in the standard canonical action, we can define the Bardeen parameter ζ [67],

$$\zeta \equiv \frac{5\rho + 3p}{3(\rho + p)}\Phi + \frac{2\rho}{3(\rho + p)}\frac{\dot{\Phi}}{H}, \quad (54)$$

and Mukhanov variable ν ,

$$u \equiv z\zeta, \quad (55)$$

where for the general action [37],

$$z \equiv \frac{a(\rho + p)^{1/2}}{c_s H}, \quad (56)$$

$$= \frac{\sqrt{2}M_{pl}a\sqrt{\epsilon}}{c_s}. \quad (57)$$

In a flat universe, after quantization, a general action still has an equation of motion similar to that for canonical actions [37]

$$\frac{d^2 u_k}{d\tau^2} + \left(c_s^2 k^2 - \frac{1}{z} \frac{d^2 z}{d\tau^2} \right) u_k = 0, \quad (58)$$

where

$$\frac{1}{z} \frac{d^2 z}{d\tau^2} = a^2 H^2 W, \quad (59)$$

with,

$$W = 2 \left[\left(1 + \frac{\eta}{2} - \kappa \right) \left(1 - \frac{\epsilon}{2} + \frac{\eta}{4} - \frac{\kappa}{2} \right) + \frac{\eta_N}{2} - \kappa_N \right] \quad (60)$$

Now inserting slow roll conditions Eq. (48), more specifically assuming $\eta \ll \frac{1}{N}$, ϵ varies very slowly, then using Eq. (42), $aH\tau(1 - \epsilon) \approx -1$, and u_k satisfies a Bessel equation,

$$\frac{d^2 u_k}{d\tau^2} + \left(c_s^2 k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2} \right) u_k = 0, \quad (61)$$

where

$$\nu^2 = \frac{W}{(1 - \epsilon)^2} + \frac{1}{4}. \quad (62)$$

In the slow roll limit Eq. (48), solution tends toward $\nu \rightarrow 3/2$. Following [37], to leading order the scalar spectral density, $\mathcal{P}_{\mathcal{R}}$ is given by

$$\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2} \left. \frac{|u_k|^2}{z^2} \right|_{c_s k = aH}, \quad (63)$$

$$\sim \frac{1}{8\pi^2 M_{pl}^2} \frac{H^2}{c_s \epsilon} \Big|_{c_s k = aH}. \quad (64)$$

The tensor spectra density to first order is

$$\mathcal{P}_h = \left. \frac{2H^2}{\pi^2 M_{pl}^2} \right|_{k=aH}. \quad (65)$$

Note that in these computations, we are implicitly assuming Bunch-Davies vacuum boundary conditions, whose validity generically has model-dependent limitations [57, 58]. Scalar perturbations are calculated at sound horizon crossing, $k_s = aH/c_s$, while tensor perturbations are fixed when $k_t = aH$, so that

$$\left. \frac{d \ln k}{d N_e} \right|_{k=k_s} = -(1 - \epsilon - \kappa), \quad (66)$$

$$\left. \frac{d \ln k}{d N_e} \right|_{k=k_t} = -(1 - \epsilon). \quad (67)$$

The scalar spectral index is given by

$$n_s - 1 \equiv \left. \frac{d \ln \mathcal{P}_{\mathcal{R}}}{d \ln k} \right|_{k=k_s}, \quad (68)$$

$$\approx -(2\epsilon + \eta + \kappa) + O(\epsilon^2, \epsilon\eta, \kappa_N, \dots), \quad (69)$$

$$n_t \equiv \left. \frac{d \ln \mathcal{P}_h}{d \ln k} \right|_{k=k_t} \approx -2\epsilon + O(\epsilon^2, \dots), \quad (70)$$

and tensor to scalar ratio

$$r \equiv \frac{\mathcal{P}_h}{\mathcal{P}_{\mathcal{R}}} \approx 16c_s \epsilon, \quad (71)$$

gives rise to the consistency relation

$$r \approx -8c_s n_t. \quad (72)$$

Note that the consistency relationship is very similar to that of multifield models. The running in the

spectral indices are given by

$$\left. \frac{dn_s}{d \ln k} \right|_{k=k_s} \approx 2\epsilon_N + \eta_N + \kappa_N, \quad (73)$$

$$\left. \frac{dn_t}{d \ln k} \right|_{k=k_t} \approx 2\epsilon_N. \quad (74)$$

As discussed in [37], the consistency relation could allow us to determine whether the action is canonical or not in the context of single field inflation. Some data fitting results exploring the effects of including c_s in Eq. (72) can be found in [59].

The primordial spectrum, as we will discuss, in theory provides information with which we might differentiate between different kinetic inflationary models in the slow roll regime defined in Eq. (48). To obtain this however, requires the measurement of both the tensor and scalar primordial spectra, including scale dependency of tensor modes, and constraints on non-Gaussianity: 1) A measurement of n_t would give a direct estimate of ϵ . 2) Comparing measurements of r and n_t in Eq. (72) gives a measure of c_s and a first insight into whether the action is canonical or not. 3) n_s allows us to constrain $2\epsilon - \eta - \kappa$, while $dn_t/d \ln k$, using Eq. (110) would constrain $2\epsilon - \eta$. Comparing these two would give a direct measure of κ . We exhaust the information coming from the two-point correlations since running in the scalar spectral index is dependent on higher order terms.

Observational tests of non-Gaussianity using the CMB provide additional measurements of the flow parameters. The CMB is most sensitive to the 3-point function of the comoving curvature perturbation, ζ ,

$$\zeta = \zeta_G - \frac{3}{5} f_{NL} \zeta_G^2, \quad (75)$$

where ζ_G is a Gaussian field and where f_{NL} gives a measure of the local intrinsic non-linearity in the curvature fluctuation as discussed in section II. Non-Gaussianity in general actions has been computed by [44, 45] with the definition of ζ having the opposite of the sign convention of Eq. (54). For generalized single field inflation, [45] finds, in the equilateral momentum triangle limit,

$$f_{NL}^{\text{equil}} \approx (-0.26 + 0.12c_s^2) \left(1 - \frac{1}{c_s^2} \right) - 0.08 \left(\frac{c_s^2}{\epsilon} \right) \frac{X^3 \mathcal{L}_{XXX}}{M_{pl}^2 H^2}, \quad (76)$$

whereas in the squeezed triangle limit [43],

$$f_{NL}^{\text{local}} \sim (n_s - 1). \quad (77)$$

Note that the f_{NL}^{local} detection recently reported in [48] uses a sign convention opposite to that used in [44, 45]. The amplitude of the primordial non-Gaussianity f_{NL} therefore could in principle be used in addition to the scalar and tensor power spectrum measurements

to obtain information about a higher derivative term, \mathcal{L}_{XXX} if one assumes single field inflation.

Current non-Gaussianity limits coming from the equilateral triangle limit of bispectrum are not competitive with the 2-point constraints, with WMAP 3-year data giving $-256 < f_{NL}^{\text{equil}} < 332$ at the 95% confidence level [60]. Prospectively the PLANCK satellite will improve this constraint with estimated errors in $\sigma(f_{NL}^{\text{equil}}) = 66.9$ at 1σ level [61]. It is however intriguing that [48] has very recently reported a positive detection of local f_{NL}^{local} . Given that this result came out after our work was completed, we leave the full discussion of its implications to a future work. A related discussion in the context of curvaton models has already appeared [62].

VI. RECONSTRUCTING THE ACTION FROM SLOW ROLL PARAMETERS

In section V we outlined how power spectrum observations can give us constraints on the slow roll parameters and c_s , which in their own right can help us differentiate broadly between theories with $c_s = 1$ from $c_s \neq 1$. In this section we take the slow roll constraints one step further and consider the question of given the constraints on these parameters and c_s , how much can be known about the original action of the inflaton field. Although we are only focusing on slow roll parameters and c_s , the formalism we discuss in this section can easily be extended to obtain more details about the original action if we include measurement of higher order correlation functions such as f_{NL} which as we explained in the last section contain higher derivative terms.

We consider the ‘ideal’ analytical reconstruction possible if $\epsilon(k)$ and $c_s(k)$ are measured over some observable range $k_{\min}(N_{e,\max}) \leq k \leq k_{\max}(N_{e,\min})$.

One can reconstruct the evolutionary trajectory for the homogenous energy density and pressure ($p = \mathcal{L}$) relative to some reference point within that range, $k_0(N_{e0})$. Combining the definition of ϵ in Eq. (40) and the conservation of energy conservation, Eq. (30), one finds

$$\frac{\rho(N_e)}{\rho(N_{e0})} = \exp \int_{N_{e0}}^{N_e} 2\epsilon(N) dN. \quad (78)$$

where $\rho(N_e)$ and $\mathcal{L}(N_e)$ are related at each time by

$$\mathcal{L}(N_e) = \rho(N_e) \left(-1 + \frac{2\epsilon(N_e)}{3} \right), \quad (79)$$

Eq. (79) and using Eq. (41) and Eq. (33) give constraints on gauge independent combinations of X and derivatives of \mathcal{L} for the on-shell trajectory,

$$X(N_e) \mathcal{L}_X(N_e) = \frac{1}{3} \epsilon(N_e) \rho(N_e) \quad (80)$$

$$X^2(N_e) \mathcal{L}_{XX}(N_e) = \frac{1}{6} \left(\frac{1}{c_s^2(N_e)} - 1 \right) \epsilon(N_e) \rho(N_e) \quad (81)$$

Notice that here the gauge ambiguity for $X(N_e)$ arises since unlike the canonical action, not having fixed the kinetic term in the action and not knowing \mathcal{L}_X as a specific function of ϕ or X to substitute for, leaves the door open to gauge ambiguities due to field redefinitions, $\phi \rightarrow \varphi \equiv f(\phi)$. We discuss the possibility of more general canonical transformations in the appendix. Typically, only $\phi \rightarrow \varphi \equiv f(\phi)$ will lead to a local field theory and more general canonical transformations will not lead to a local theory. Hence in order to establish the evolution more specifically we must choose a scalar field gauge. This can, for example, be done by choosing $\mathcal{L}_X(N_e) = c_s^{-1}(N_e)$, as is the case for canonical and DBI inflation, leading to

$$X(N_e) = \frac{1}{3}\epsilon(N_e)c_s(N_e)\rho(N_e) \quad (82)$$

$$\Delta\phi \equiv \phi(N_e) - \phi(N_{e0}) = - \int_{N_{e0}}^{N_e} \sqrt{2\epsilon c_s} dN, \quad (83)$$

where for simplicity in notation for the rest of this section only, we are using the $M_{pl} = 1$ convention (geometrized units). An alternative useful gauge is taking $X(N_e) = 1/2$, for which,

$$\Delta\phi = - \int_{N_{e0}}^{N_e} \frac{dN}{H} = - \int_{N_{e0}}^{N_e} dN \sqrt{\frac{3}{\rho(N_e)}}, \quad (84)$$

Notice that here we are aiming to reconstruct a two dimensional manifold $\mathcal{L}(X, \phi)$ in a three dimensional space (\mathcal{L}, X, ϕ) and we have so far shown that after fixing the gauge ambiguity, the one dimensional trajectory of $\mathcal{L}(\frac{1}{2}, \phi)$ is required to lie on this manifold and locally minimize it at the same time, however in the X direction only the first and second derivatives are constrained leaving the higher derivatives along X completely free. Therefore any action consistent with such observations automatically equates to satisfying the above boundary conditions. One can easily find all such manifolds of $\mathcal{L}(X, \phi)$ by solving an arbitrary third order differential equation along characteristic curves of $(\mathcal{L}, X, \phi = \text{constant})$ obeying the boundary conditions at $(\frac{1}{2}, \phi)$. Any action consistent with the constraints on $X\mathcal{L}_X$ and $X^2\mathcal{L}_{XX}$ can then be written in the form

$$\begin{aligned} \tilde{\mathcal{L}} = & q(X, \phi) + \mathcal{L}\left(\frac{1}{2}, \phi\right) - q\left(\frac{1}{2}, \phi\right) \\ & + \left[\mathcal{L}_X\left(\frac{1}{2}, \phi\right) - q_X\left(\frac{1}{2}, \phi\right) \right] \left(X - \frac{1}{2}\right) \\ & + \frac{1}{2} \left[\mathcal{L}_{XX}\left(\frac{1}{2}, \phi\right) - q_{XX}\left(\frac{1}{2}, \phi\right) \right] \left(X - \frac{1}{2}\right)^2 \end{aligned} \quad (85)$$

where q is an arbitrary function of ϕ and X . It is also straight forward to show that the trajectory of $\tilde{\mathcal{L}}(\frac{1}{2}, \phi)$ is minimizing the action, since equation of motion Eq. (32) for $\tilde{\mathcal{L}}$ at $X = 1/2$ simplifies to,

$$\mathcal{L}_\phi\left(\frac{1}{2}, \phi\right) - \mathcal{L}_{X\phi}\left(\frac{1}{2}, \phi\right) = 3H(N_e)\mathcal{L}_X\left(\frac{1}{2}, \phi\right) \quad (86)$$

which, using Eq. (80), turns up to be simply an alternative way of writing $\rho_N = 2\epsilon\rho$ which has already been set to remain valid. This can be seen more clearly through the following example. Lets consider the case where $\epsilon \sim \frac{1}{2N_e} \ll 1$ where we are taking N_e to be decreasing during inflation. This is what one would expect for a quadratic potential in the case of a canonical action $c_s = 1$. We will consider the implications for the action if c_s deviates slightly from one, $c_s = 1 - \delta$. We first obtain H using Eq. (78):

$$H = H_1 \exp\left(\int_1^{N_e} \frac{dN}{2N}\right) = H_1 N_e^{1/2} \quad (87)$$

where $H_1 = H|_{N_e=1}$. Now fixing the gauge to $X = 1/2$ we get

$$\frac{d\phi}{dN} = \frac{-1}{H_1 N_e^{1/2}} \Rightarrow \phi = -2 \frac{N_e^{1/2}}{H_1} \quad (88)$$

the above equation combined with Eq. (80) and Eq. (81) yield

$$\mathcal{L}\left(\frac{1}{2}, \phi\right) = H_1^2(1 - H_1^2\phi^2) \quad (89)$$

$$\mathcal{L}_X\left(\frac{1}{2}, \phi\right) = H_1^2 \quad (90)$$

$$\mathcal{L}_{XX}\left(\frac{1}{2}, \phi\right) \sim 2H_1^2\delta \quad (91)$$

Now substituting these result in Eq. (85), and for instance taking $q = 0$, in the limit of $\epsilon \ll 1$ or equivalently $|H_1\phi| \gg 1$ the action will have a following form:

$$\tilde{\mathcal{L}}_1(X, \phi) \sim H_1^2 \left[-\frac{3}{4}(H_1\phi)^2 + X + \delta X^2 \right], \quad (92)$$

which after a field redefinition is slightly deviated from the a canonical action with quadratic potential. However if we take $q = \lambda X^3$ then the action will be:

$$\begin{aligned} \tilde{\mathcal{L}}_2(X, \phi) = & \tilde{\mathcal{L}}_1(X, \phi) \\ & + \lambda \left[X^3 - \frac{1}{8} - \frac{3}{4}\left(X - \frac{1}{2}\right) - \frac{3}{2}\left(X - \frac{1}{2}\right)^2 \right] \end{aligned} \quad (93)$$

which also satisfies the equation of motion at $X = \frac{1}{2}$ and fits ϵ and c_s regardless of the magnitude of λ .

VII. INFLATIONARY FLOW EQUATIONS

In the previous section it was shown that, even after fixing a gauge, there are an infinite number of different actions that can match the same observation, however it is possible to write down one dynamical evolution for the on-shell trajectory for all of them. That is to say, just like the canonical case, we can

obtain $H(\phi)$ or the function $\mathcal{L}(\phi)$ on the solution trajectory but, unlike before where it would be equated to a unique potential $V(\phi)$, it will not correspond to a unique $\mathcal{L}(X, \phi)$.

All the gauge invariant parameters that we have introduced so far belong to two categories: first, combinations of H and its derivatives with respect to e-folding number N_e :

$$\begin{aligned} H \quad , \quad \epsilon &= \frac{d \ln H}{d N_e}, \\ \eta &= -\frac{d \ln \epsilon}{d N_e} = \frac{1}{\epsilon} \frac{d^2}{d N_e^2} \ln H, \quad \eta_N, \quad \eta_{NN}, \dots \end{aligned} \quad (94)$$

and second, combinations of c_s and its derivatives with respect to N :

$$c_s, \quad \kappa = \frac{1}{c_s} \frac{d c_s}{d N_e}, \quad \kappa_N, \dots \quad (95)$$

By truncating these parameters at some derivative order to zero and then setting initial values for rest of them at $N_e = N_{e0}$ one could approximate H or c_s with Taylor expansions in terms of N_e up to a convergence radius N_{max} .

An alternative approach using *inflationary flow equations* to describe an action beyond the slow-roll assumption has been used extensively for canonical inflation [49, 55, 63] and DBI inflation [41]. Here we discuss how this formalism can be extended to a general action and a general gauge.

The inflationary flow equations are used to derive a Taylor expansion of H , \mathcal{L} , and c_s and other gauge invariant quantities in terms of a specific choice of scalar field, ϕ , for example

$$\begin{aligned} H(\phi) &= H_0 + M_{pl} H'_0 \left(\frac{\Delta \phi}{M_{pl}} \right) + \dots \\ &+ \frac{1}{(l+1)!} M_{pl}^{l+1} H_0^{[l+1]} \left(\frac{\Delta \phi}{M_{pl}} \right)^{l+1} + \dots \end{aligned} \quad (96)$$

and hence the coefficients have the nontrivial terms in the form of ,

$$Q_l(H)|_{\phi_0} = \left[\left(\frac{d N_e}{d \phi} \frac{d}{d N_e} \right)^l H \right]_{\phi_0} \quad (97)$$

and similarly terms of the form $Q_l(c_s)$ for the Taylor expansion of c_s . Since X and \mathcal{L}_X are not invariant under the field redefinition and

$$\frac{d N_e}{d \phi} = \pm \frac{H}{\sqrt{2X}} = \pm \left(\frac{\mathcal{L}_X}{2\epsilon} \right)^{1/2}, \quad (98)$$

the $Q_l(H)$ and $Q_l(c_s)$ are in general gauge dependent. Fixing a gauge, as is done in DBI and canonical inflation with $\mathcal{L}_X = c_s^{-1}$, sets this dependency.

For a general gauge, we can write the gauge invariant slow roll parameters as

$$\epsilon = \frac{2M_{pl}^2}{\mathcal{L}_X} \left(\frac{H'}{H} \right)^2, \quad (99)$$

$$\kappa = \frac{2M_{pl}^2}{\mathcal{L}_X} \left(\frac{H'}{H} \frac{(c_s^{-1})'}{c_s^{-1}} \right), \quad (100)$$

and introduce gauge dependent parameters

$$\tilde{\eta} \equiv \frac{2M_{pl}^2}{\mathcal{L}_X} \left(\frac{H''}{H} \right), \quad (101)$$

$$\tilde{\kappa} \equiv \frac{2M_{pl}^2}{\mathcal{L}_X} \left(\frac{H'}{H} \frac{\mathcal{L}_X'}{\mathcal{L}_X} \right). \quad (102)$$

where in canonical and DBI inflation the gauge choice leads to $\tilde{\kappa} = \kappa$.

$\tilde{\eta}$ and $\tilde{\kappa}$ are not invariant under a redefinition of the scalar field $\phi \rightarrow \varphi(\phi)$,

$$\tilde{\eta} = -\frac{\dot{X}}{2HX} - \frac{\dot{\mathcal{L}}_X}{H\mathcal{L}_X}, \quad (103)$$

$$\tilde{\kappa} = -\frac{\dot{\mathcal{L}}_X}{H\mathcal{L}_X}, \quad (104)$$

however, the combination $2\tilde{\eta} - \tilde{\kappa}$ is invariant under the transformation,

$$2\tilde{\eta} - \tilde{\kappa} = 2\epsilon - \eta = -\frac{(X\dot{\mathcal{L}}_X)}{HX\mathcal{L}_X}. \quad (105)$$

The inflationary flow equation hierarchy is obtained by defining three sets of variables,

$${}^l\lambda(\phi) \equiv \left(\frac{2M_{pl}^2}{\mathcal{L}_X} \right)^l \left(\frac{H'}{H} \right)^{l-1} \frac{H^{[l+1]}}{H} \quad (106)$$

$${}^l\alpha(\phi) \equiv \left(\frac{2M_{pl}^2}{\mathcal{L}_X} \right)^l \left(\frac{H'}{H} \right)^{l-1} \frac{(c_s^{-1})^{[l+1]}}{c_s^{-1}} \quad (107)$$

$${}^l\beta(\phi) \equiv \left(\frac{2M_{pl}^2}{\mathcal{L}_X} \right)^l \left(\frac{H'}{H} \right)^{l-1} \frac{\mathcal{L}_X^{[l+1]}}{\mathcal{L}_X} \quad (108)$$

for $l \geq 1$, where $H^{[l+1]} \equiv d^{l+1}H/d\phi^{l+1}$ and $\tilde{\eta} = {}^1\lambda$. For DBI inflation and canonical inflation ${}^l\alpha = {}^l\beta$. As explained above, in general ${}^l\lambda$, ${}^l\alpha$ and ${}^l\beta$ are not invariant under scalar field redefinitions.

Noting that

$$\frac{d\phi}{dN_e} = \frac{2M_{pl}^2}{\mathcal{L}_X} \frac{H'}{H} \quad (109)$$

the evolutionary paths of these parameters simplify to coupled first order differential equations with respect to N_e . Then,

$$\epsilon_N = -\epsilon(2\epsilon - 2\tilde{\eta} + \tilde{\kappa}) = -\epsilon\eta, \quad (110)$$

$$\tilde{\eta}_N = -\tilde{\eta}(\epsilon + \tilde{\kappa}) + {}^2\lambda, \quad (111)$$

$$\kappa_N = -\kappa(\epsilon - \tilde{\eta} + \tilde{\kappa} + \kappa) + \epsilon^1\alpha, \quad (112)$$

$$\tilde{\kappa}_N = -\tilde{\kappa}(\epsilon - \tilde{\eta} + 2\tilde{\kappa}) + \epsilon^1\beta, \quad (113)$$

and for $l \geq 1$,

$${}^l\lambda_N = -{}^l\lambda[l\epsilon - (l-1)\tilde{\eta} + l\tilde{\kappa}] + {}^{l+1}\lambda, \quad (114)$$

$${}^l\alpha_N = -{}^l\alpha[(l-1)\epsilon - (l-1)\tilde{\eta} + l\tilde{\kappa} + \kappa] + {}^{l+1}\alpha, \quad (115)$$

$${}^l\beta_N = -{}^l\beta[(l-1)\epsilon - (l-1)\tilde{\eta} + (l+1)\tilde{\kappa}] + {}^{l+1}\beta. \quad (116)$$

Following the nomenclature of [49], the Taylor expansion of the Hubble factor, c_s^{-1} and \mathcal{L}_X in powers of ϕ can be written,

$$H(\phi) = H_0 \left[1 + A_1 \left(\frac{\Delta\phi}{M_{pl}} \right) + \dots + A_{M_A+1} \left(\frac{\Delta\phi}{M_{pl}} \right)^{M_A+1} + \dots \right], \quad (117)$$

$$c_s^{-1}(\phi) = c_{s0}^{-1} \left[1 + B_1 \left(\frac{\Delta\phi}{M_{pl}} \right) + \dots + B_{M_B+1} \left(\frac{\Delta\phi}{M_{pl}} \right)^{M_B+1} + \dots \right], \quad (118)$$

$$\mathcal{L}_X(\phi) = \mathcal{L}_{X0} \left[1 + C_1 \left(\frac{\Delta\phi}{M_{pl}} \right) + \dots + C_{M_C+1} \left(\frac{\Delta\phi}{M_{pl}} \right)^{M_C+1} + \dots \right], \quad (119)$$

where

$$A_l \equiv \frac{1}{l!} \frac{M_{pl}^l}{H_0} H^{[l+1]} \Big|_{\phi=\phi_0} \quad (120)$$

$$B_l \equiv \frac{1}{l!} \frac{M_{pl}^l}{c_{s0}^{-1}} (c_s^{-1})^{[l+1]} \Big|_{\phi=\phi_0} \quad (121)$$

$$C_l \equiv \frac{1}{l!} \frac{M_{pl}^l}{\mathcal{L}_{X0}} \mathcal{L}_X^{[l+1]} \Big|_{\phi=\phi_0} \quad (122)$$

and H_0 , \mathcal{L}_{X0} and c_{s0} are the values of H , \mathcal{L}_X and c_s at the reference point $\phi_0 \equiv \phi(N_{e0})$ with $\Delta\phi \equiv \phi(N_e) - \phi(N_{e0})$. Note that for scenarios such as relativistic DBI where c_s^{-1} diverges at desiter limit and Taylor expansion description is invalid out of the convergence radius of ϕ_0 , instead one could use the Taylor expansion of c_s and \mathcal{L}_X^{-1} .

Using Eq. (35) and Eq. (114) - Eq. (115),

$$A_1 = \sqrt{\frac{\epsilon_0 \mathcal{L}_{X0}}{2}}, \quad (123)$$

$$A_{l+1} = \frac{\mathcal{L}_{X0}^{l+1}}{2^l(l+1)!A_1^{l-1}} {}^l\lambda_0, \quad (124)$$

$$B_1 = \frac{\kappa_0 \mathcal{L}_{X0}}{2A_1}, \quad (125)$$

$$B_{l+1} = \frac{\mathcal{L}_{X0}^{l+1}}{2^l(l+1)!A_1^{l-1}} {}^l\alpha_0. \quad (126)$$

$$C_1 = \frac{\tilde{\kappa}_0 \mathcal{L}_{X0}}{2A_1}, \quad (127)$$

$$C_{l+1} = \frac{\mathcal{L}_{X0}^{l+1}}{2^l(l+1)!A_1^{l-1}} {}^l\beta_0, \quad (128)$$

The flow equations derived in this section apply to inflationary models independent of whether inflation is slow roll or not. Often in applying the flow equations, however, it is commonly assumed that within the chosen gauge, the series are convergent, and the hierarchies in ${}^l\lambda$, ${}^l\alpha$ and ${}^l\beta$ can be truncated with non-zero values for a finite range of l , $l \leq M_A$, $l \leq M_B$ and $l \leq M_C$ respectively in the chosen gauge.

VIII. CONCLUSIONS

Complementary CMB and large scale structure measurements over scales spanning four orders of magnitude have driven impressive improvements in the measurement of the primordial scalar power spectrum. In addition improvements in non-Gaussianity constraints are expected from the PLANCK satellite and there is the exciting prospect of tensor mode measurements in the near future, with a number of CMB surveys being developed to target B-mode polarization. With the hope of connecting this to high energy theory, there has been significant interest in establishing what current and planned observations might elucidate about the primordial spectrum of fluctuations from inflation, in terms of potential reconstruction for canonical inflation and action reconstruction in specific theories such as DBI inflation.

In this paper we extend these considerations to address what we can maximally learn about the inflationary action without making any assumptions, a priori, about its form. We establish how observational constraints on the inflationary slow roll parameters could be successfully applied to reconstruct the general action over observable scales in the context of single field inflationary models. Under the assumption of slow roll inflation, we have demonstrated that in an idealized case in which $\{H, c_s, \epsilon, \eta, \kappa\}$ are measured over a finite range of scales, we analytically obtain the trajectory of the general action $\mathcal{L}, X\mathcal{L}_X, X^2\mathcal{L}_{XX}$, independent of the scalar field definition, with respect to some reference point. With the specification of a gauge, the measurement of the first level of flow parameters enables trajectories of X, \mathcal{L}_X and \mathcal{L}_{XX} and information about \mathcal{L}_ϕ and $\mathcal{L}_{X\phi}$ to be established.

Using the Hamilton-Jacobi formalism, we extend the inflationary flow parameter approach to describe the evolutionary trajectories of general actions. This involves introducing three hierarchies of flow parameters to describe the evolution of a general action without using the specific gauge, $\mathcal{L}_X = c_s^{-1}$, used in canonical and DBI inflation. These equations hold for all single field inflationary scenarios, whether or not slow roll conditions are met.

Observations promise to allow us to reconstruct a wealth of information about the general action including powerful insights into the form of the Lagrangian kinetic, potential and hybrid terms and the relative importance of kinetic and potential compo-

nents over the course of the trajectory. As it is difficult to obtain large observable non-Gaussianities without non-minimal kinetic terms (and/or resorting to curvaton scenarios), future detection of non-Gaussianities would make formalisms such as the one we present here indispensable to understand what kind of high energy theories are compatible with cosmological data. This is good news for PLANCK and other future CMB experiments which are certain to obtain increasingly precise data regarding non-Gaussianities, and it is also good news for high energy theorists looking for distinctive clues to the identity of the inflaton. In work in preparation, we are investigating the observational constraints on the general inflationary action using this formalism.

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APPENDIX A: CANONICAL TRANSFORMATIONS VERSUS FIELD REDEFINITIONS

Although a local field redefinition of the form $\phi = \phi(\tilde{\phi})$ yields a classical Lagrangian density which describes the same physics, such transformations only form a subset of the canonical transformations $\phi = \phi(\tilde{\phi}, \tilde{\pi})$ and $\pi = \pi(\tilde{\phi}, \tilde{\pi})$ which by construction preserves the physics. In this Appendix, it is shown how the Lagrangian transforms under a more general set of canonical transformations.

Restricting to 0+1 dimensions, we find the interesting result that a minimal kinetic term can be transformed into a system with non-minimal kinetic term (non-minimal here is to be distinguished from non-canonical since the latter is a straightforward transformation). As a byproduct, we find an exact solution to the non-linear differential equation Eq. (A50) through the use of a canonical transformation. Unfortunately, the generalization of such minimal to non-minimal transitioning systems to 3+1 dimensions results in a non-local Lagrangian.

To begin with, let us show that a local field redefinition of the form $\phi = g(\tilde{\phi})$ leads to a physically

equivalent equation of motion. Consider a Lagrangian density of the form $\mathcal{L}(X, \phi)$ where $X \equiv (\partial\phi)^2$. Taking the variation of the action

$$S = \int d^4x \mathcal{L}(X, \phi) \quad (\text{A1})$$

yields the EOM

$$2\partial_\mu \left\{ \partial^\mu \phi \frac{\partial}{\partial X} \mathcal{L}(X, \phi) \right\} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (\text{A2})$$

Define the local field redefinition

$$\phi = g(\tilde{\phi}). \quad (\text{A3})$$

We then have

$$\begin{aligned} & 2g'(\tilde{\phi})\partial_\mu \left\{ \frac{1}{g'(\tilde{\phi})} \right\} \partial^\mu \tilde{\phi} \frac{\partial \mathcal{L}((g'(\tilde{\phi}))^2 \tilde{X}, g(\tilde{\phi}))}{\partial \tilde{X}} - \frac{\partial \mathcal{L}}{\partial \tilde{\phi}} \\ & + 2\partial_\mu \left\{ \partial^\mu \tilde{\phi} \frac{\partial \mathcal{L}((g'(\tilde{\phi}))^2 \tilde{X}, g(\tilde{\phi}))}{\partial \tilde{X}} \right\} + 2g'' \frac{\tilde{X}}{g'} \frac{\partial \mathcal{L}}{\partial \tilde{X}} = 0. \end{aligned} \quad (\text{A4})$$

Because the first and the last terms cancel, we end up with an equation of motion for a new Lagrangian of the form

$$\tilde{\mathcal{L}}(\tilde{X}, \tilde{\phi}) = \mathcal{L}((g'(\tilde{\phi}))^2 \tilde{X}, g(\tilde{\phi})). \quad (\text{A5})$$

The stress energy tensor for the new Lagrangian density can also be checked to be physically identical to the original:

$$\tilde{X} = g^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \quad (\text{A6})$$

$$S = \int d^4x \sqrt{g} \tilde{\mathcal{L}} \quad (\text{A7})$$

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}. \quad (\text{A8})$$

Hence, it is clear that a local field redefinition leads to the same physics. Now, let us consider the more general possibility of a canonical transformation.

Restrict to the 0+1 dimension inflaton theory, which would correspond to a classical mechanics problem in one spatial dimension. A transformation from the phase space variable $\{\phi, p\}$ to $\{\tilde{\phi}, \tilde{p}\}$

$$\phi = \phi(\tilde{\phi}, \tilde{p}; t) \quad (\text{A9})$$

$$p = p(\tilde{\phi}, \tilde{p}; t) \quad (\text{A10})$$

corresponds to a canonical transformation if Hamilton's equations are preserved, which in turn implies

$$p\dot{\phi} - H(\phi, p, t) = \tilde{p}\dot{\tilde{\phi}} - \tilde{H}(\tilde{\phi}, \tilde{p}, t) + \frac{d}{dt} F(\phi, \tilde{\phi}, t) \quad (\text{A11})$$

for some function \tilde{H} and F . The function $F(\phi, \tilde{\phi}, t)$ is called the generating function for the canonical transformation. The canonical transformation generated by F is then

$$p = \frac{\partial}{\partial \tilde{\phi}} F(\phi, \tilde{\phi}, t) \quad (\text{A12})$$

$$-\tilde{p} = \frac{\partial}{\partial \phi} F(\phi, \tilde{\phi}, t) \quad (\text{A13})$$

with the new Hamiltonian given by

$$\tilde{H}(\tilde{\phi}, \tilde{p}, t) = H(\phi, p, t) + \frac{\partial}{\partial t} F(\phi, \tilde{\phi}, t). \quad (\text{A14})$$

Since the Lagrangian is a Legendre transformation of the Hamiltonian, we have

$$\tilde{L}(\tilde{\phi}, \dot{\tilde{\phi}}; t) = \tilde{p}\dot{\tilde{\phi}} - \tilde{H}(\tilde{\phi}, \tilde{p}, t) \quad (\text{A15})$$

where

$$\dot{\tilde{\phi}} = \frac{\partial \tilde{H}}{\partial \tilde{p}}. \quad (\text{A16})$$

This is the new Lagrangian generated by a canonical transformation, which contains the same physics. For example, as long as the canonical transformation is accomplished in a time independent manner, the energy density remains the same since $H = \tilde{H}$ according to Eq. (A14).

One may try to express \tilde{L} more directly in terms of F by formally solving some of the algebraic relations above, but as we will see the final result is not that illuminating except for seeing how the generating function explicitly mixes ϕ and p in the field redefinition. Start with the Hamiltonian after the canonical transformation written as

$$\begin{aligned} \tilde{H}(\tilde{\phi}, \tilde{p}, t) = & H\left(\phi = \phi_*(\tilde{\phi}, \tilde{p}, t), \frac{\partial}{\partial \tilde{\phi}} F(\phi, \tilde{\phi}, t)|_{\phi=\phi_*(\tilde{\phi}, \tilde{p}, t)}, t\right) \\ & + \frac{\partial}{\partial t} F(\phi = \phi_*(\tilde{\phi}, \tilde{p}, t), \tilde{\phi}, t) \end{aligned} \quad (\text{A17})$$

where ϕ_* solves the equation

$$-\tilde{p} = \frac{\partial}{\partial \tilde{\phi}} F(\phi, \tilde{\phi}, t)|_{\phi=\phi_*}. \quad (\text{A18})$$

Note that this amounts to a field redefinition involving both ϕ and p . Hence, the Lagrangian becomes

$$\tilde{L}(\tilde{\phi}, \dot{\tilde{\phi}}; t) = \tilde{p}\dot{\tilde{\phi}} - \tilde{H}(\tilde{\phi}, \tilde{p}, t) \quad (\text{A19})$$

where \tilde{p} is eliminated by solving the equation

$$\dot{\tilde{\phi}} = \frac{\partial \tilde{H}}{\partial \tilde{p}} \quad (\text{A20})$$

$$\begin{aligned} &= \frac{\partial}{\partial \tilde{p}} \left[H(\phi = \phi_*(\tilde{\phi}, \tilde{p}, t), \frac{\partial}{\partial \tilde{\phi}} F(\phi, \tilde{\phi}, t)|_{\phi=\phi_*(\tilde{\phi}, \tilde{p}, t)}, t) \right. \\ &\quad \left. + \frac{\partial}{\partial t} F(\phi = \phi_*(\tilde{\phi}, \tilde{p}, t), \tilde{\phi}, t) \right] \end{aligned} \quad (\text{A21})$$

Unfortunately, there does not seem to be an elucidating general simplification for \tilde{L} . Hence, we turn to some explicit examples.

Consider the original Lagrangian to be

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 \quad (\text{A22})$$

and the generating function

$$F(\phi, \tilde{\phi}, t) = \phi\tilde{\phi}^2. \quad (\text{A23})$$

The Hamiltonian can be obtained as follows:

$$p = \dot{\phi} \quad (\text{A24})$$

$$H(\phi, p) = p\dot{\phi} - L \quad (\text{A25})$$

$$= \frac{1}{2}(p^2 + m^2\phi^2) \quad (\text{A26})$$

$$p = \dot{\phi}^2 \quad (\text{A27})$$

$$-\tilde{p} = 2\phi\tilde{\phi} \quad (\text{A28})$$

$$\tilde{H} = \frac{1}{2}(p^2 + m^2\phi^2) \quad (\text{A29})$$

$$= \frac{1}{2}\left(\frac{m^2}{4\phi^2}\tilde{p}^2 + \tilde{\phi}^4\right) \quad (\text{A30})$$

Hence, we have

$$\dot{\tilde{\phi}} = \frac{m^2}{4\phi^2}\tilde{p} \quad (\text{A31})$$

$$\tilde{L} = \tilde{p}\dot{\tilde{\phi}} - \frac{1}{2}\left(\frac{m^2}{4\phi^2}\tilde{p}^2 + \tilde{\phi}^4\right) \quad (\text{A32})$$

$$= \frac{1}{m^2}[2\tilde{\phi}^2\dot{\tilde{\phi}}^2 - \frac{m^2}{2}\tilde{\phi}^4] \quad (\text{A33})$$

In this case, the field redefinition $\phi = \tilde{\phi}^2$ would have generated the equivalent Lagrangian.

Next, we consider an example in which a non-minimal kinetic term can be transformed into a minimal kinetic term. Consider the original Lagrangian to be

$$L = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (\text{A34})$$

and the generating function

$$F(\phi, \tilde{\phi}, t) = \phi\tilde{\phi}^2 + \tilde{\phi}^3. \quad (\text{A35})$$

The Hamiltonian can be obtained as follows:

$$p = \dot{\phi} \quad (\text{A36})$$

$$H(\phi, p) = \frac{1}{2}p^2 + V(\phi) \quad (\text{A37})$$

$$p = \tilde{\phi}^2 \quad (\text{A38})$$

$$-\tilde{p} = 2\phi\tilde{\phi} + 3\tilde{\phi}^2 \quad (\text{A39})$$

$$\tilde{H} = \frac{1}{2}\tilde{\phi}^4 + V\left(\frac{-\tilde{p} - 3\tilde{\phi}^2}{2\tilde{\phi}}\right) \quad (\text{A40})$$

Hence, we have

$$\dot{\tilde{\phi}} = \frac{-1}{2\tilde{\phi}}V'\left(\frac{-\tilde{p} - 3\tilde{\phi}^2}{2\tilde{\phi}}\right) \quad (\text{A41})$$

which allows us to express \tilde{p} in terms of $\tilde{\phi}$ and $\dot{\tilde{\phi}}$:

$$\tilde{p} = -2\tilde{\phi}V'^{-1}(-2\tilde{\phi}\dot{\tilde{\phi}}) - 3\tilde{\phi}^2 \quad (\text{A42})$$

Hence, our Lagrangian becomes

$$\tilde{L} = \tilde{p}\dot{\tilde{\phi}} - \left[\frac{1}{2}\tilde{\phi}^4 + V\left(\frac{-\tilde{p} - 3\tilde{\phi}^2}{2\tilde{\phi}}\right) \right] \quad (\text{A43})$$

$$= -2\tilde{\phi}\dot{\tilde{\phi}}V'^{-1}(-2\tilde{\phi}\dot{\tilde{\phi}}) - \frac{d}{dt}\tilde{\phi}^3 - \left[\frac{1}{2}\tilde{\phi}^4 + V(V'^{-1}(-2\tilde{\phi}\dot{\tilde{\phi}})) \right] \quad (\text{A44})$$

Suppose we consider $V = \frac{1}{4}\lambda\phi^4$. We would find

$$V'(\phi) = \lambda\phi^3 \quad (\text{A45})$$

giving

$$V'^{-1}(-2\tilde{\phi}\dot{\tilde{\phi}}) = \frac{1}{\lambda^{1/3}}(-2\tilde{\phi}\dot{\tilde{\phi}})^{1/3} \quad (\text{A46})$$

$$V(V'^{-1}(-2\tilde{\phi}\dot{\tilde{\phi}})) = \frac{1}{4}\frac{1}{\lambda^{1/3}}(-2\tilde{\phi}\dot{\tilde{\phi}})^{4/3} \quad (\text{A47})$$

$$\tilde{L} = \frac{3}{4}\frac{1}{\lambda^{1/3}}(-2\tilde{\phi}\dot{\tilde{\phi}})^{4/3} - \frac{d}{dt}\tilde{\phi}^3 - \frac{1}{2}\tilde{\phi}^4 \quad (\text{A48})$$

Hence, the interesting point of this example is that a canonical transformation has turned an analytic kinetic term into a non-analytic one.

Let's check that the equation of motion generated from this Lagrangian can give the same solution as the original Lagrangian. The equation of motion with this Lagrangian is

$$\tilde{\phi}\ddot{\tilde{\phi}} - 3(2^{-1/3})\lambda^{1/3}\tilde{\phi}^{8/3}\dot{\tilde{\phi}}^{2/3} - \dot{\tilde{\phi}}^2 = 0. \quad (\text{A49})$$

To compare to the solutions of the original equation,

$$\ddot{\phi} + \lambda\phi^3 = 0 \quad (\text{A50})$$

we need to consider an observable and a boundary condition. Since we are looking at Minkowski physics,

we can simply choose the energy density to be the observable. As far as mapping the boundary conditions are concerned, note that Eqs. (A38) and (A36) imply

$$\tilde{\phi}^2(0) = \dot{\phi}(0) \quad (\text{A51})$$

$$2\tilde{\phi}\dot{\tilde{\phi}} = \ddot{\phi}(0) = -\lambda\phi^3(0). \quad (\text{A52})$$

Now, the solution to the original variable equation Eq. (A50) with the boundary condition

$$\phi(t=0) = 0 \quad (\text{A53})$$

$$\dot{\phi}(t=0) = A \quad (\text{A54})$$

has a solution

$$\phi(t) = At[1 - \frac{\lambda}{20}A^2t^4 + \mathcal{O}(\lambda^2A^4t^8)]. \quad (\text{A55})$$

To compare, using Eqs. (A51) and (A52), we should solve Eq. (A49) with the boundary conditions

$$\tilde{\phi}(0) = \sqrt{A} \quad (\text{A56})$$

$$\dot{\tilde{\phi}}(0) = 0. \quad (\text{A57})$$

We see that in fact, in the $\tilde{\phi}$ variables,

$$\tilde{\phi}(t) = \sqrt{A} \quad (\text{A58})$$

is an exact solution satisfying the desired boundary conditions.

The stress energy tensors to compare are then

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{\lambda}{4}\phi^4 \quad (\text{A59})$$

and

$$\tilde{T}_{00} = \frac{1}{2}\tilde{\phi}^4 + \frac{1}{4}\frac{1}{\lambda^{1/3}}(-2\tilde{\phi}\dot{\tilde{\phi}})^{4/3} \quad (\text{A60})$$

Inserting Eq. (A55) into Eq. (A59), we obtain

$$T_{00} \approx \frac{A^2}{2} + \mathcal{O}(t^8) \quad (\text{A61})$$

where if we had not solved the equation of motion, we would have had a t^4 term on the right hand side. On the other hand, inserting Eq. (A58) into Eq. (A60), we obtain

$$\tilde{T}_{00} = \frac{A^2}{2} \quad (\text{A62})$$

exactly. Hence, we have given a non-trivial check that the solution arising from the non-minimal kinetic Lagrangian of Eq. (A48) gives the exactly the same observable as the solution arising from the minimal kinetic Lagrangian of Eq. (A34) with $V(\phi) = \frac{\lambda}{4}\phi^4$.

Thus far, we had been working in 0+1 dimensions (i.e. the spatial variation of the field had been frozen). Let us consider how this generalizes to field theory. Unfortunately, we will show that the interesting example of minimal kinetic term leading to a nonminimal kinetic term requires a non-local transformation. First, we would like to show that

$$\phi = \phi(\tilde{\phi}, \tilde{\pi}) \quad (\text{A63})$$

$$\pi = \pi(\tilde{\phi}, \tilde{\pi}) \quad (\text{A64})$$

can be generated by the generating function $F(\phi, \tilde{\phi})$ with the new Hamiltonian given by

$$\tilde{\mathcal{H}}(\tilde{\phi}, \tilde{\pi}) = \mathcal{H}(\phi, \pi) \quad (\text{A65})$$

$$\pi = \frac{\partial}{\partial \phi} F(\phi, \tilde{\phi}) \quad (\text{A66})$$

$$-\tilde{\pi} = \frac{\partial}{\partial \tilde{\phi}} F(\phi, \tilde{\phi}). \quad (\text{A67})$$

To begin, take the total time derivative of F :

$$\frac{d}{dt} F(\phi, \tilde{\phi}) = \frac{\partial F}{\partial \phi} \dot{\phi} + \frac{\partial F}{\partial \tilde{\phi}} \dot{\tilde{\phi}}. \quad (\text{A68})$$

Using this with Eqs. (A66) and (A67), we have

$$\tilde{\pi} \dot{\tilde{\phi}} + \frac{d}{dt} F(\phi, \tilde{\phi}) = \pi \dot{\phi}. \quad (\text{A69})$$

Next, using Eq. (A65), we find

$$\tilde{\pi} \dot{\tilde{\phi}} - \tilde{\mathcal{H}}(\tilde{\phi}, \tilde{\pi}) + \frac{d}{dt} F(\phi, \tilde{\phi}) = \pi \dot{\phi} - \mathcal{H}(\phi, \pi), \quad (\text{A70})$$

which says that the two Lagrangian densities are identical up to a total time derivative. Note that the total derivative can be non-trivial when it involves ϕ and $\tilde{\phi}$ and not just ϕ or $\tilde{\phi}$.

To obtain the new Lagrangian, we use

$$\dot{\phi}(x) = \frac{\delta}{\delta \tilde{\pi}(x)} \int d^3x \tilde{\mathcal{H}} \quad (\text{A71})$$

to solve for $\tilde{\pi}(x)$. Unfortunately, as we will now show, this method generally fails to produce a local Lagrangian since solving Eq. (A71) for $\tilde{\pi}$ generically involves solving an elliptic PDE. To see this, start with

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi). \quad (\text{A72})$$

The Hamiltonian can be obtained as follows:

$$\pi = \dot{\phi} \quad (\text{A73})$$

$$\mathcal{H}(\phi, p) = \frac{1}{2}[\pi^2 + (\nabla\phi)^2] + V(\phi) \quad (\text{A74})$$

$$F(\phi, \phi) = \phi\dot{\phi}^2 + \tilde{\phi}^3 \quad (\text{A75})$$

$$\pi = \tilde{\phi}^2 \quad (\text{A76})$$

$$-\tilde{\pi} = 2\phi\tilde{\phi} + 3\tilde{\phi}^2 \quad (\text{A77})$$

$$\tilde{\mathcal{H}} = \frac{1}{2}[\tilde{\phi}^4 + (\nabla[\frac{-\tilde{\pi} - 3\tilde{\phi}^2}{2\tilde{\phi}}])^2] + V(\frac{-\tilde{\pi} - 3\tilde{\phi}^2}{2\tilde{\phi}}) \quad (\text{A78})$$

Hence, we have

$$\dot{\phi} = \frac{\delta}{\delta \tilde{\pi}} \int d^3x \tilde{\mathcal{H}} = \frac{-1}{2\tilde{\phi}} V'(\frac{-\tilde{\pi} - 3\tilde{\phi}^2}{2\tilde{\phi}}) - \frac{1}{2\tilde{\phi}} \nabla^2 [\frac{\tilde{\pi} + 3\tilde{\phi}^2}{2\tilde{\phi}}] \quad (\text{A79})$$

which allows us to express $\tilde{\pi}$ in terms of $\tilde{\phi}$ and $\dot{\tilde{\phi}}$, but only at the expense of giving up locality (i.e. one must solve an elliptic PDE). This is the main qualitative difference between Lagrangian densities obtained from the more general canonical transformations and local field redefinitions.

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- [66] For a systematic diagrammatic approach to computing correlation functions in the δN formalism, see [64].
- [67] Note that the sign of ζ here is the opposite of the sign convention used in Section II.